

## Amplitude equations and fast transition to chaos in rings of coupled oscillators

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We study the coupling induced destabilization in an array of identical oscillators coupled in a ring structure where the number of oscillators in the ring is large. The coupling structure includes different types of interactions with several next neighbors. We derive an amplitude equation of Ginzburg-Landau type, which describes the destabilization of a uniform stationary state in a ring with a large number of nodes. Applying these results to unidirectionally coupled Duffing oscillators, we explain the phenomenon of a fast transition to chaos, which has been numerically observed in such systems. More specifically, the transition to chaos occurs on an interval of a generic control parameter that scales as the inverse square of the size of the ring, i.e. for sufficiently large system, we observe practically an immediate transition to chaos.

The understanding of the dynamical behavior of networks of coupled oscillators can contribute to the explanation of various collective phenomena that can be observed in coupled systems in biology, economy, or physics<sup>1-3</sup>. Discrete media, e.g. in neural systems, can exhibit complex coupling structures including feed-forward loops, large coupling ranges, and interaction mechanisms of different kind. In such systems, the influence of the coupling can transform the simple dynamics of a single oscillator into complicated spatio-temporal structures of the network dynamics. Amplitude equations of Ginzburg-Landau type have been a powerful tool for a universal description of spatio-temporal dynamics and pattern formation in continuous media. In this paper, we apply this technique to a large class of coupled oscillator systems with a ring structure, where we assume that the number of oscillators in the ring is large. In addition, we use the amplitude equation together with the corresponding scaling laws to explain the coupling induced fast transition from homogeneous stationary behavior to high dimensional chaos, which can be observed in certain coupled oscillator systems.

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## I. INTRODUCTION

Networks of coupled oscillators have been the subject of extensive research in the last decade. Coupled systems can display a huge variety of dynamical phenomena, starting from synchronization phenomena in various types of inhomogeneous or irregular networks, up to complex collective behavior, such as for example various forms of phase transitions, traveling waves<sup>4-6</sup>, phase-locked patterns, amplitude death states<sup>7</sup>, or so called chimera states that display a regular pattern of coherent and incoherent motion<sup>8-11</sup>. Of particular interest are situations, where complex spatio-temporal structures can emerge in regular arrays of identical units induced only by the coupling interaction. In many cases, the resulting phenomena differ substantially from corresponding situations in continuous media<sup>12</sup> and depend strongly on the underlying network topology.

Our specific interest is in the emergence of spatio-temporal structures in a homogeneous

array of identical units that have a stable uniform equilibrium at which the coupling vanishes. As a classical paradigm, the Turing instability gives an example of a coupling induced instability in such a setting. This phenomenon has of course a direct counterpart in the discrete setting, but it turns out that there appear some genuinely new phenomena. In<sup>13–15</sup> it has been shown that also in a ring of unidirectionally coupled oscillators, i.e. in a purely convective setting, the Eckhaus scenario of coexisting diffusive patterns can be observed. In<sup>16</sup> it has been shown that Duffing oscillators coupled in the same way, exhibit a complex transition to spatio-temporal chaos. In this paper we develop a general theoretical framework for such phenomena in large arrays. We derive an amplitude equation of Ginzburg-Landau type that governs the local dynamics close to the destabilization of the uniform steady state. It resembles several features that are already well known in the context of reaction-diffusion systems<sup>17,18</sup>. But in contrast to these results, it can be applied to much more general coupling mechanisms, including also the case of unidirectional and anti-diffusive interaction and allows also for a mixture of such interactions with several next neighbors. Such an interplay of attractive and repulsive coupling is an essential feature of neural systems. As a specific feature, the convective part will appear in the amplitude equation as a rotation of the coordinates in an intermediate time scale that is faster than the diffusive processes described by the Ginzburg-Landau equation.

Having deduced the amplitude equation and the corresponding scaling laws in terms of the number of oscillators, which is assumed to tend to infinity, we use this theory for the explanation of a specific phenomenon that has been reported in<sup>16</sup> for a ring of unidirectionally coupled Duffing oscillators. There, it has been shown numerically that for a large number of oscillators  $N$ , there is an almost immediate transition from homogeneous stationary behavior to high-dimensional chaos. Based on our amplitude equation, we argue that in such systems, one can expect generically that such a transition occurs within a parameter interval of the size  $1/N^2$ . We consider a generic case, where the control parameter enters already the linear parts of the dynamical equations, e.g. a diffusive coupling strength. Finally, we demonstrate this phenomenon by a numerical example, where we also evaluate the scaling behavior of the parameter interval where the transition to chaos takes place for an increasing number of oscillators.

## II. MODEL EQUATION, SPECTRAL CONDITIONS, AND NOTATIONS

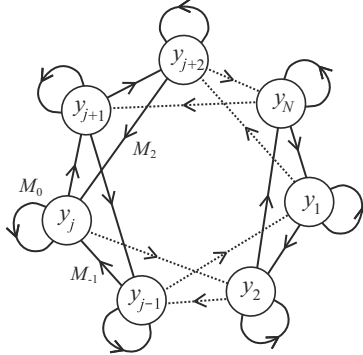


Figure II.1. An example of a ring of  $N$  coupled oscillators. Apart from the self-coupling  $M_0$ , each oscillator  $y_j$  is also coupled with  $y_{j+2}$  ( $M_2$ ) as well as  $y_{j-1}$  ( $M_{-1}$ ). See Eq. (II.1) for the equation of motion.

We are interested in a system of  $N$  identical coupled oscillators that has an uniform equilibrium, where the coupling vanishes. The coupling network is organized in a ring structure, where interactions of several next neighbors are possible. Such systems can be written in general form as

$$\dot{y}_j = \sum_m M_m(p) y_{j+m} + h(y_j, y_{j+1}, \dots, y_{j+N}; p), \quad (\text{II.1})$$

where  $y_j \in \mathbb{R}^n$ ,  $j = 1, \dots, N$  describes the state of the  $j$ -th oscillator. The ring structure is induced by periodic boundary conditions, i.e. all indexes have to be considered modulo  $N$ . The linear part of the dynamics is given by the  $n \times n$  matrices  $M_m(p)$ ,  $m = 1, \dots, N$ , depending on the bifurcation parameter  $p$ , which account for the coupling to the  $m$ -th neighbor; in particular  $M_0(p)$  describes the linear part of the local dynamics (self-coupling). The nonlinear part  $h$ , again including a local dependence and a dependence on the  $m$ -th neighbor, should vanish at the origin  $h(0, \dots, 0; p) = 0$  and have also zero derivatives there. Note that this system is symmetric (equivariant) with respect to index shift. Figure II.1 illustrates an example with self coupling and coupling to the neighbor on the left and to the second neighbor on the right. The specific form of (II.1) also implies that the coupling vanishes at the equilibrium  $y_1 = \dots = y_N = 0$ , which is true e.g. when the coupling is a function of the difference  $y_j - y_m$  for any two coupled oscillators  $j, m$ .

The characteristic equation for the linearization at the zero equilibrium of (II.1) can be factorized as

$$\chi(p, \lambda, e^{i2\pi j/N}) = \det \left[ \lambda \text{Id} - \sum_m e^{im2\pi j/N} M_m(p) \right] = 0,$$

where Id denotes the identity matrix in  $\mathbb{R}^n$  and the index  $j = 1, 2, \dots, N$  accounts for the  $N$ -th roots of unity that appear as the eigenvalues of the circular coupling structure<sup>19</sup>. Following the approach in<sup>13,14</sup>, we replace for large  $N$  the discrete numbers  $2\pi j/N$  by a continuous parameter  $\varphi$ , and obtain the *asymptotic continuous spectrum*

$$\Lambda_p = \left\{ \lambda \in \mathbb{C} : \chi(p, \lambda, e^{i\varphi}) = \det \left[ \lambda \text{Id} - \sum_m e^{im\varphi} M_m(p) \right] = 0, \varphi \in [0, 2\pi) \right\}, \quad (\text{II.2})$$

which contains all eigenvalues and which for large  $N$  is covered densely by the eigenvalues. Since the expression (II.2) is periodic in  $\varphi$ , the asymptotic continuous spectrum  $\Lambda_p$  has generically the form of one or several closed curves  $\lambda_p(\varphi)$  in the complex plane, parametrized by  $\varphi$ .

At the bifurcation value  $p = 0$ , we assume that the asymptotic continuous spectrum touches the imaginary axis at at some point  $i\omega_0$  (see Fig. II.2), i.e. the following conditions are fulfilled

$$\lambda_0(\varphi_0) = i\omega_0, \quad \frac{\partial \lambda_0}{\partial \varphi}(\varphi_0) = i\kappa_1, \quad \kappa_1 \in \mathbb{R}. \quad (\text{II.3})$$

The first condition from (II.3) means that the point  $i\omega_0$  belongs to the asymptotic continuous spectrum, while the second condition from (II.3) guarantees that the real part  $\Re(\lambda(\varphi))$  is tangent to the zero axis at  $\varphi = \varphi_0$  (see Fig. II.2). This tangency condition is quite natural and describes the condition for the destabilization (or bifurcation) of the zero solution a large ring of coupled oscillators. Indeed, if the spectrum  $\Lambda_p$  is contained in the left half of the complex plane with  $\Re \lambda < 0$ , then the uniform equilibrium  $y_1 = \dots = y_N = 0$  is asymptotically stable. As soon as  $\Lambda_p$  crosses the imaginary axis, it becomes unstable for sufficiently large  $N$ .

Before we present our main result, we now introduce some useful notations and formulate a technical lemma, which follows from the bifurcation conditions (II.3). With  $v_0$  and  $v_1$  we denote the eigenvector and the adjoint eigenvector to the critical eigenvalue  $\lambda_0(\varphi_0) = i\omega_0$ , which we assumed in (II.3) to exist for  $p = 0$ . Moreover, it will be convenient to denote by

$$L_0 = \sum_m e^{im\varphi_0} M_m(0),$$

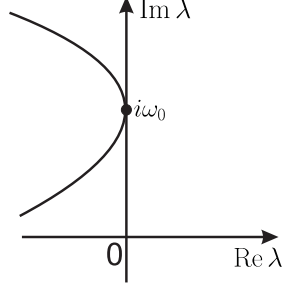


Figure II.2. Asymptotic continuous spectrum  $\Lambda_p$  at the destabilization; schematically.

$$L_1 = \sum_m m e^{im\varphi_0} M_m(0),$$

$$L_2 = \sum_m m^2 e^{im\varphi_0} M_m(0)$$

the “moments” of the coupling matrices  $M_m$ . In this notation, the equations for the eigenvectors  $v_0$  and  $v_1$  read as

$$[i\omega_0 \text{Id} - L_0] v_0 = 0, \quad (\text{II.4})$$

$$[-i\omega_0 \text{Id} - L_0^*] v_1 = 0. \quad (\text{II.5})$$

These vectors can be normalized as

$$|v_0|^2 = 1, \quad \langle v_0, v_1 \rangle = 1. \quad (\text{II.6})$$

Finally, we expand the matrices  $M_m(p)$  with respect to the parameter  $p$  and write them as  $M_m(0) + pK_m + \mathcal{O}(p^2)$ . In this way, we can define

$$L_K = \sum e^{im\varphi_0} K_m.$$

**Lemma 1.** *Assume that  $\varphi_0$  is a regular point of the asymptotic continuous spectrum (II.2), such that  $\lambda_0(\varphi)$  exists and is locally differentiable in a small neighborhood of  $\varphi_0$ . Further, let the bifurcation condition (II.3) hold. Then*

$$\langle L_1 v_0, v_1 \rangle = \kappa_1, \quad (\text{II.7})$$

The proof of this Lemma will be given together with the proof of the main result in Section IV.

### III. MAIN RESULT: REDUCTION TO GINZBURG-LANDAU EQUATION

In this section, we present an amplitude equation that describes for system (II.1) the dynamics close to the destabilization threshold in the limit of large  $N$ . We perform a limiting procedure, where we assume that a fixed number  $R$  of neighbors is involved in the coupling while the total number of oscillators  $N$  tends to infinity. Also the nonlinearity  $h$  is assumed to depend on  $R$  next neighbors only. Hence, in the sequel all summation will extend over  $m \in \{-R, \dots, R\}$  and the nonlinearity will be written as  $h(y_{j-R}, \dots, y_{j+R}; p)$ , independently on  $N$ . In this way, the coupling becomes local in the limit  $N \rightarrow \infty$ , and coupling terms will be approximated by derivatives of the amplitude. Consequently, our results will be valid for large rings, where the coupling range is small compared to the total size. It will be shown that the amplitude equation has the form of a complex Ginzburg-Landau equation with periodic boundary conditions. The derivation is quite technical and we give here in Proposition 2 only the main assertions. The proof is deferred to Section IV.

**Proposition 2.** *Assume that the bifurcation condition (II.3) holds and  $\varphi_0$  is a regular point of the asymptotic continuous spectrum  $\Lambda_0$ . Additionally, let the points  $\pm 3i\omega_0$  not belong to the asymptotic continuous spectrum with  $\varphi = \varphi_0$  (nonresonance condition). Let the nonlinearity  $h$  be of third order. Introduce the small parameter*

$$\varepsilon = \frac{1}{N}$$

*and apply the multiple scale ansatz*

$$y_j(t) = \varepsilon e^{i\omega_0 t + i\varphi_0 j} v_0 A(T_1, x_1, T_2) + \varepsilon^3 e^{3i(\omega_0 t + \varphi_0 j)} v_2 A^3(T_1, x_1, T_2) + c.c., \quad (\text{III.1})$$

*with the amplitude  $A \in \mathbb{C}$  depending on the rescaled coordinates  $T_1 = \varepsilon t$ ,  $T_2 = \varepsilon^2 t$ , and  $x_1 = \varepsilon j$  (c.c. denotes complex conjugated terms,  $v_2 \in \mathbb{C}^n$ ) to the following system*

$$\dot{y}_j = \sum_{m=-R}^R (M_m(0) + \varepsilon^2 r K_m + \mathcal{O}(\varepsilon^4)) y_{j+m} + h(y_{j-R}, \dots, y_{j+R}; p), \quad (\text{III.2})$$

*with the rescaled parameter  $p = \varepsilon^2 r$  and  $j = 1, \dots, N$  with periodic boundary conditions. Then, the solvability conditions up to the order  $\varepsilon^3$  imply the following partial differential equation of Ginzburg-Landau type*

$$\partial_{T_2} u = r \kappa_2 u + \frac{\kappa_3}{2} \partial_{\xi}^2 u + \zeta u |u|^2 \quad (\text{III.3})$$

with periodic boundary conditions

$$u(\xi, T_2) = u(\xi + 1, T_2),$$

where  $u(\xi, T_2)$  with  $\xi \in [0, 1]$  is related to the amplitude  $A$  by

$$A(T_1, x_1, T_2) = u(\kappa_1 T_1 + x_1, T_2). \quad (\text{III.4})$$

The coefficient  $\kappa_1$  is given by (II.3) or (II.7), and

$$\kappa_2 = \langle L_K v_0, v_1 \rangle, \quad \kappa_3 = \langle L_2 v_0, v_1 \rangle.$$

Finally, the coefficient of the nonlinearity  $\zeta$  and the vector  $v_2 \in \mathbb{C}^n$  have to be determined by the nonlinearity  $h$  according to (IV.1) and (IV.2).

According to this proposition, small solutions of a coupled system of the form (III.2) that has a parameter close to criticality, i.e.  $p = O(\varepsilon^2)$  can be approximated in the form (III.1), where the amplitude  $A$  is related to a solution of the Ginzburg-Landau equation (III.3) via (III.4). Note that the relation (III.4) introduces a rotating frame on the timescale  $T_1 = \varepsilon t$  with rotation velocity  $\kappa_1$ . The time evolution of the Ginzburg-Landau equation enters only on the slowest time scale  $T_2 = \varepsilon^2 t$ .

*Remark 3.* For the case of a symmetric coupling ( $M_k = M_{-k}$ ) the amplitude equation (III.3) has all real coefficients  $\kappa_1$ ,  $\kappa_2$  and  $\kappa_3$  and the rotation velocity vanishes  $\kappa_1 = 0$ .

## IV. PROOFS

**Proof of Lemma 1.** We have to show that  $\kappa_1$ , defined by (II.3), can be calculated as given in (II.7). To this end, we differentiate the eigenvalue equation (cf. (II.4))

$$[\lambda_0(\varphi) \text{Id} - L_0] v(\varphi) = 0,$$

with respect to  $\varphi$ :

$$\left[ \text{Id} \frac{\partial}{\partial \varphi} \lambda_0(\varphi) - i \sum_m m e^{im\varphi} M_m(0) \right] v(\varphi) + \left[ \lambda_0(\varphi) \text{Id} - \sum_m e^{im\varphi} M_m(0) \right] \frac{\partial}{\partial \varphi} v(\varphi) = 0.$$

Evaluating the obtained expression at  $\varphi = \varphi_0$ ,  $\lambda(\varphi_0) = i\omega_0$ , and  $v(\varphi_0) = v_0$ , we obtain

$$[\text{Id} i\kappa_1 - iL_1] v_0 + [i\omega_0 \text{Id} - L_0] \frac{\partial v}{\partial \varphi}(\varphi_0) = 0.$$



The projection onto  $v_0$  gives

$$i\kappa_1 \langle v_0, v_1 \rangle - i \langle L_1 v_0, v_1 \rangle + \left\langle [i\omega_0 \text{Id} - L_0] \frac{\partial v}{\partial \varphi}(\varphi_0), v_1 \right\rangle = 0,$$

which implies

$$i\kappa_1 - i \langle L_1 v_0, v_1 \rangle + \left\langle \frac{\partial v}{\partial \varphi}(\varphi_0) v_0, [-i\omega_0 \text{Id} - L_0^*] v_1 \right\rangle = 0.$$

Taking into account (II.5), we obtain the relation (II.7), which proves the Lemma.

**Proof of Proposition 2.** Substituting the multiple scale ansatz (III.1) into (III.2), we obtain

$$\begin{aligned} & \varepsilon \frac{d}{dt} \left( e^{i\omega_0 t + i\varphi_0 j} v_0 A + \varepsilon^2 e^{3i(\omega_0 t + \varphi_0 j)} v_2 A^3 + c.c. \right) \\ &= \varepsilon \sum_m \left( M_m(0) + \varepsilon^2 r K_m \right) e^{i\omega_0 t + i\varphi_0 j} e^{im\varphi_0} v_0 A(T_1, x_1 + m\varepsilon, T_2) \\ &+ \varepsilon^3 \sum_m \left( M_m(0) + \varepsilon^2 r K_m \right) e^{3i(\omega_0 t + \varphi_0 j)} e^{3im\varphi_0} v_2 A^3(T_1, x_1 + m\varepsilon, T_2) + c.c. \\ &+ h(y_{j-R}, \dots, y_{j+R}; \varepsilon^2 r) \end{aligned}$$

Dividing the obtained equation by  $\varepsilon$ , expanding necessary arguments of  $A$ , we obtain up to the terms of the order  $\varepsilon^2$  (the complex conjugated terms are omitted here for brevity)

$$\begin{aligned} & e^{i\omega_0 t + i\varphi_0 j} v_0 \left( i\omega_0 A + \varepsilon \partial_{T_1} A + \varepsilon^2 \partial_{T_2} A \right) + 3i\omega_0 \varepsilon^2 e^{3i(\omega_0 t + \varphi_0 j)} v_2 A^3 \\ &= e^{i\omega_0 t + i\varphi_0 j} \sum_m \left( M_m(0) + \varepsilon^2 r K_m \right) e^{im\varphi_0} v_0 \left( A + m\varepsilon \partial_{x_1} A + \frac{1}{2} m^2 \varepsilon^2 \partial_{x_1}^2 A \right) \\ &+ \varepsilon^2 \sum_m M_m(0) e^{3i(\omega_0 t + \varphi_0 j)} e^{3im\varphi_0} v_2 A^3 \\ &+ \varepsilon^2 e^{i\omega_0 t + i\varphi_0 j} A |A|^2 h_1(v_0) + \varepsilon^2 e^{3i(\omega_0 t + \varphi_0 j)} A^3 h_2(v_0) \end{aligned}$$

where  $h_1(v_0)$  and  $h_2(v_0)$  are determined by the leading terms in the expansion of the non-linearity. Note that due to our assumption,  $h$  is of third order and the leading order terms are given by expanding  $h(y_{j-R}, \dots, y_{j+R}; 0)$  in the homogeneous state  $y_m = \alpha v_0 + c.c.$ ,  $m = j - R, \dots, j + R$  with respect to  $\alpha \in \mathbb{C}$ . The solvability condition requires that different harmonics as well as different orders of  $\varepsilon$  up to  $\varepsilon^2$  are equal. We start with the first harmonic. By equating the terms containing  $e^{i\omega_0 t + i\varphi_0 j}$  we obtain the following equation

$$v_0 \left( i\omega_0 A + \varepsilon \partial_{T_1} A + \varepsilon^2 \partial_{T_2} A \right) =$$

$$= \sum_m (M_m(0) + \varepsilon^2 r K_m) e^{im\varphi_0} v_0 \left( A + m\varepsilon \partial_{x_1} A + \frac{1}{2} m^2 \varepsilon^2 \partial_{x_1}^2 A \right) + \varepsilon^2 A |A|^2 h_1(v_0).$$

Since it should be satisfied for all  $\varepsilon$ , we first consider the  $\varepsilon^0$  equation

$$i\omega_0 A v_0 = A L_0 v_0,$$

which holds according to the spectral condition (II.4). The  $\varepsilon^1$  terms result into

$$v_0 \partial_{T_1} A - L_1 v_0 \partial_{x_1} A = 0.$$

Multiplication with  $v_1^T$  and using (II.7) from Lemma 1, we obtain

$$\partial_{T_1} A - \kappa_1 \partial_{x_1} A = 0.$$

This will be accounted for by introducing the new amplitude  $u$  by

$$u(\xi, T_2) = u(\kappa_1 T_1 + x_1, T_2) = A(T_1, x_1, T_2)$$

in a correspondingly rotating coordinate  $\xi = \kappa_1 T_1 + x_1$ . Finally, the  $\varepsilon^2$  terms result into

$$v_0 \partial_{T_2} A = r A L_K v_0 + \frac{1}{2} \partial_{x_1}^2 A L_2 v_0 + A |A|^2 h_1(v_0)$$

Note that the dependence on  $T_1$  does not show up in this equation. Hence, after multiplication with  $v_1^T$ , we can write it in terms of  $u$  as

$$\partial_{T_2} u = r \kappa_2 u + \frac{\kappa_3}{2} \partial_\xi^2 u + \zeta u |u|^2,$$

where

$$\zeta = \langle h_1(v_0), v_1 \rangle. \quad (\text{IV.1})$$

Finally, it is simple to check that the solvability of the terms for the third harmonic leads to the expression

$$v_2 = \left[ 3i\omega_0 - \sum_m M_m(0) e^{3im\varphi_0} \right]^{-1} h_2(v_0) \quad (\text{IV.2})$$

Here, the existence of a nonzero solution  $v_2$  is guaranteed by the nonresonance condition. Indeed, since the points  $\pm 3i\omega_0$  do not belong to the asymptotic continuous spectrum with  $\varphi = \varphi_0$ , the matrix  $[3i\omega_0 - \sum_m M_m(0) e^{3im\varphi_0}]$  is non-singular. Finally, note that the set of equations should be complemented by periodic boundary conditions in  $\xi$ , taking into account the ring structure of system (III.2). The proposition is proved.

## V. APPLICATION: SCALING OF THE BIFURCATION TRANSITION

It is well known that in the complex Ginzburg-Landau equation (III.3) a destabilization transition is possible, where in a series of bifurcations a homogeneous stationary state loses its stability and, via periodic and quasiperiodic motion, a regime of spatio-temporal chaos appears<sup>20–22</sup>. We will now use the result about the amplitude equation in order to trace back this destabilization transition to the original coupled oscillator system of the form (II.1). Specific attention will be paid to the scaling behavior of the parameter region where this transition takes place.

Starting with the scale free amplitude equation

$$\partial_{T_2} u = r\kappa_2 u + \frac{\kappa_3}{2} \partial_\xi^2 u + \zeta u |u|^2 \quad (\text{V.1})$$

with periodic boundary conditions, we assume that at  $r = r_0$  a destabilization of the homogeneous stationary state takes place and for further increasing parameter  $r$  a transition to spatio-temporal chaos can be observed. This transition takes place within some bounded interval  $\Delta r$ . Considering now the corresponding behavior of an oscillator system that is described by this amplitude equation, we have to introduce the scalings given in Proposition 2. In particular, according to the parameter scaling  $p = r\varepsilon^2$ , the same transition takes place in the coupled oscillator system within a parameter interval

$$\Delta p(N) = \Delta r \varepsilon^2 = \Delta r \frac{1}{N^2},$$

of the corresponding parameter  $p$  of the oscillator system. Therefore, one can expect that a generic transition to chaos (or hyperchaos) in the system of  $N$  coupled oscillators occurs within a parameter interval that scales as  $1/N^2$ . In the following section, we present an example where the assumptions that we made above are satisfied and, consequently, in the oscillator system an almost immediate transition from stationary behavior to chaos can be observed.

## VI. EXAMPLE: A RING OF UNIDIRECTIONALLY COUPLED DUFFING OSCILLATORS

In the previous section we discussed the scaling law for the transition to chaos in a ring of coupled oscillators. In this section we provide a numerical example to illustrate this scaling.

We use the autonomous Duffing oscillator described by the following second order ordinary differential equation

$$\ddot{y} + d\dot{y} + ay + y^3 = 0, \quad (\text{VI.1})$$

where  $d$  and  $a$  are positive constants. System (VI.1) is a single-well Duffing oscillator, which has a single equilibrium point at  $y = \dot{y} = 0$ . Due to the presence of damping ( $d > 0$ ) this equilibrium is an attractor for all initial conditions. We consider now a ring of  $N$  such oscillators with a linear unidirectional coupling to the next neighbor. Introducing new coordinates  $x = y$ ,  $z = \dot{y}$  and the coupling into Eq. (VI.1), the equations of motion have the form

$$\begin{aligned} \dot{x}_j &= z_j, \\ \dot{z}_j &= -dz_j - ax_j - x_j^3 + k(x_{j+1} - x_j), \end{aligned} \quad (\text{VI.2})$$

where  $k$  is the coupling coefficient and indices are considered modulo  $N$ . A detailed analysis of the behavior of this system was already presented in<sup>14</sup>. For at least three coupled oscillators and increasing coupling strength  $k$ , one can observe rich dynamics starting from periodic oscillations to hyperchaos. Here we focus our attention on the transition to chaos (hyperchaos) and its dependence on the number of oscillators. In our numerics, we used the fixed parameter values  $a = 0.1$  and  $d = 0.3$ . We calculated the maximum Lyapunov exponents for increasing  $N$  and varying coupling coefficient  $k$ . based on this, we determined two values:  $k_H$  where the uniform stationary state loses its stability in a Hopf bifurcation, and  $k_{Ch}$  where the transition to chaos takes place. A reliable computation of the Lyapunov exponents was possible only for moderate values of  $N$  ( $N < 30$ ). For larger systems, we monitored the behavior in an appropriately chosen Poincare section in order to determine  $k_H$  and  $k_{Ch}$ . Fig. VI.1(a) shows that for increasing  $N$  the distance between  $k_H$  and  $k_{Ch}$  decreases. Finally, for a large enough number of oscillators, the transition to chaos appears practically immediately after the Hopf bifurcation. In order to validate our conjecture about the  $1/N^2$  scaling, we plot in Fig. VI.1(b) the scaled transition intervals

$$k_{Re} = (k_{Ch} - k_H) N^2.$$

The numerical results indicate that the scaled transition interval neither tends to zero nor diverges to infinity, which supports the scaling assumption.

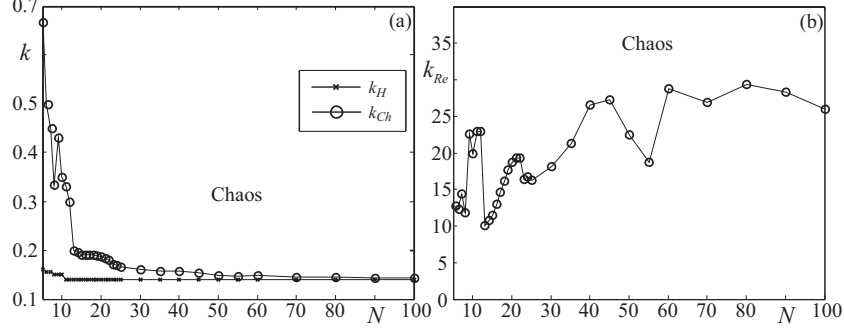


Figure VI.1. (a) Values of the coupling parameter  $k_H$  at Hopf bifurcation (crosses) and  $k_{Ch}$  at the transition to chaos (circles) for increasing number of oscillators  $N$ . (b) Rescaled transition interval  $k_{Re}$  (circles) versus  $N$ .

In Fig. VI.2 we present the Lyapunov spectrum for thirty coupled oscillators. Panel (a) shows that at  $k_{Ch} = 0.164$  two Lyapunov exponents become positive practically simultaneously. A similar behavior has been observed for other large values of  $N$ . This shows that the transition to chaos and to hyperchaos occurs in a large ring almost simultaneously. With further increasing coupling coefficient  $k$  (Fig. VI.2(b)) one can observe that more and more Lyapunov exponents become positive which leads to a high dimensional chaos<sup>23,24</sup>.

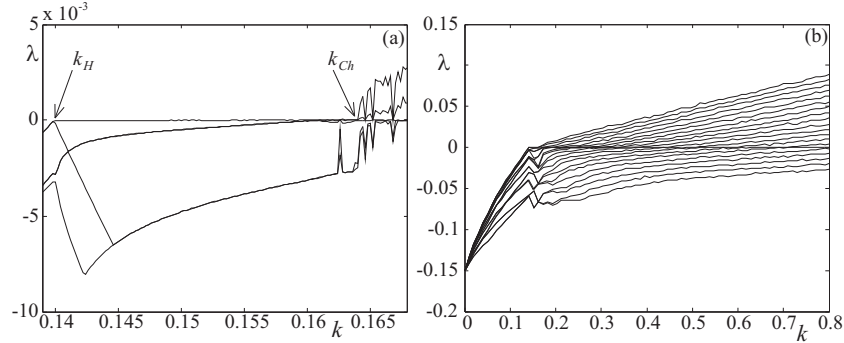


Figure VI.2. Lyapunov exponents for  $N = 30$  unidirectionally coupled Duffing oscillators (a) five largest Lyapunov exponents,  $k_H$  and  $k_{Ch}$  indicate the Hopf bifurcation and the transition to chaos,  $k \in (0.139, 0.17)$ , (b) twenty largest Lyapunov exponents for  $k \in (0.0, 0.8)$ . At  $k = 0.8$ , there are fourteen positive Lyapunov exponents.

## VII. CONCLUSIONS

The results of this paper are twofold. The first one is more theoretical: we have derived an amplitude equation for small amplitude solutions of a ring system of coupled oscillators. This equation has the form of a complex Ginzburg-Landau equation with periodic boundary conditions. Such an amplitude equation is typical for reaction-diffusion systems<sup>17,18</sup> in continuous media and can also be found for systems with large delay<sup>25</sup>. Obviously, a similar behavior should be expected in discrete systems with diffusive coupling. However, our results show that in coupled oscillator systems the Ginzburg-Landau equation can also be used to describe the dynamics in systems with unidirectional, i.e. purely convective coupling or even with anti-diffusive interaction. In this sense, the class of coupled oscillators that we treated in this paper differs substantially from discrete analogs of the classical results for continuous media.

For the second result, we applied the amplitude equation to the specific scenario of the transition to spatio-temporal chaos in such systems. In this way, we provided a general framework for the coupling induced transition to high dimensional chaos in coupled oscillator systems, that has been described for a specific example in<sup>14</sup>. In particular, we were able to show that the observed fast transition to chaos is a generic feature and follows a  $1/N^2$  scaling law for systems with a large number  $N$  of oscillators. As a result of this scaling, one observes for large systems a practically immediate transition from a uniform stationary state to chaos. We illustrated this behavior by a numerical example, where the emergence of high dimensional chaos is demonstrated by corresponding Lyapunov spectra.

## ACKNOWLEDGMENTS

S. Yanchuk acknowledges the support of the DFG collaborative research center "Control of self-organizing nonlinear systems: Theoretical methods and concepts of application" (SFB910) under the project A3. P. Perlikowski, A. Stefański and T. Kapitaniak acknowledge the support from Foundation for Polish Science - Team Programme (Project No TEAM/2010-5/5) and P. Perlikowski START Programme.

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